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Complete graphs whose topological symmetry groups are polyhedral

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We determine for which m the complete graph K_m has an embedding in S^3 whose topological symmetry group is isomorphic to one of the polyhedral groups A_4, A_5 or S_4.

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1 Introduction

Characterizing the symmetries of a molecule is an important step in predicting its chemical behavior. Chemists have long used the group of rigid symmetries, known as the *point group*, as a means of representing the symmetries of a molecule. However, molecules which are flexible or partially flexible may have symmetries which are not included in the point group. Jon Simon [10] introduced the concept of the *topological symmetry group* in order to study symmetries of such nonrigid molecules. The topological symmetry group provides a way to classify not only the symmetries of molecular graphs, but the symmetries of any graph embedded in S^3.

We define the topological symmetry group as follows. Let γ be an abstract graph, and let $\mathrm{Aut}(\gamma)$ denote the automorphism group of γ. Let Γ be the image of an embedding of γ in S^3. The *topological symmetry group* of Γ, denoted by $\mathrm{TSG}(\Gamma)$, is the subgroup of $\mathrm{Aut}(\gamma)$ which is induced by homeomorphisms of the pair (S^3, Γ). The *orientation preserving topological symmetry group* of Γ, denoted by $\mathrm{TSG}_+(\Gamma)$, is the subgroup of $\mathrm{Aut}(\gamma)$ which is induced by orientation preserving homeomorphisms of the pair (S^3, Γ). In this paper we are only concerned with $\mathrm{TSG}_+(\Gamma)$, and thus for simplicity we abuse notation and refer to the group $\mathrm{TSG}_+(\Gamma)$ simply as the *topological symmetry group* of Γ.

Frucht [8] showed that every finite group is the automorphism group of some connected graph. Since every graph can be embedded in S^3, it is natural to ask whether every finite group can be realized as $\mathrm{TSG}_+(\Gamma)$ for some connected graph Γ embedded

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in S^3. Flapan, Naimi, Pommersheim and Tamvakis proved in [6] that the answer to this question is "no", and proved that there are strong restrictions on which groups can occur as topological symmetry groups. For example, it was shown that $\mathrm{TSG}_+(\Gamma)$ can never be the alternating group A_n for $n > 5$.

The special case of topological symmetry groups of complete graphs is interesting to consider because a complete graph K_n has the largest automorphism group of any graph with n vertices. In [7], Flapan, Naimi and Tamvakis characterized which finite groups can occur as topological symmetry groups of embeddings of complete graphs in S^3 as follows.

Complete Graph Theorem [7] *A finite group H is isomorphic to $\mathrm{TSG}_+(\Gamma)$ for some embedding Γ of a complete graph in S^3 if and only if H is a finite cyclic group, a dihedral group, a subgroup of $D_m \times D_m$ for some odd m or A_4, S_4 or A_5.*

We use D_m to denote the dihedral group with $2m$ elements. The groups A_4, S_4 or A_5, are known as *polyhedral groups* because they consist of the group of rotations of a tetrahedron (which is isomorphic to A_4), the group of rotations of a cube or octahedron (which is isomorphic to S_4) and the group of rotations of a dodecahedron or icosahedron (which is isomorphic to A_5).

Observe that the Complete Graph Theorem does not tell us which complete graphs can have a given group H as their topological symmetry group. In this paper we characterize which complete graphs can have each of the polyhedral groups as its topological symmetry group. In particular, in the following results we determine for which m the graph K_m has an embedding Γ with $\mathrm{TSG}_+(\Gamma) \cong A_4$, A_5 or S_4.

A_4 Theorem *A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that $\mathrm{TSG}_+(\Gamma) \cong A_4$ if and only if $m \equiv 0, 1, 4, 5, 8 \pmod{12}$.*

A_5 Theorem *A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that $\mathrm{TSG}_+(\Gamma) \cong A_5$ if and only if $m \equiv 0, 1, 5, 20 \pmod{60}$.*

S_4 Theorem *A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that $\mathrm{TSG}_+(\Gamma) \cong S_4$ if and only if $m \equiv 0, 4, 8, 12, 20 \pmod{24}$.*

Observe that if K_n has an embedding with topological symmetry group isomorphic to A_5 or S_4, then K_n also has an embedding with topological symmetry group isomorphic to A_4.

In [5] we characterize which complete graphs can have a cyclic group, a dihedral group or another subgroup of $D_m \times D_m$ as its topological symmetry group.

2 Necessity of the conditions

In this section, we prove the necessity of the conditions given in the A_4, A_5 and S_4 Theorems. We begin by listing some results that were proved elsewhere that will be useful to us.

Orbits Lemma (Chambers–Flapan–O'Brien [3]) *If α and β are permutations of a finite set such that α and β commute, then β takes α–orbits to α–orbits of the same length.*

D_2 Lemma [3] *If $m \equiv 3 \pmod 4$, then there is no embedding Γ of K_m in S^3 such that $\mathrm{TSG}_+(\Gamma)$ contains a subgroup isomorphic to D_2.*

Recall that the groups A_4 and A_5 can be realized as the group of rotations of a solid tetrahedron and a solid dodecahedron respectively. Looking at each of these groups of rotations we see that any two cyclic subgroups of the same order are conjugate. The group S_4 can be realized as the group of rotations of a cube. It follows that all cyclic groups of order 3 or order 4 are conjugate. Up to conjugacy, S_4 contains two cyclic groups of order 2, those which are contained in A_4 and those which are not. This implies the following observation that we will make use of in this section.

Fixed Vertex Property *Let $G \cong A_4$, A_5 and suppose G acts faithfully on a graph Γ. Then all elements of G of a given order fix the same number of vertices. Furthermore, since all of the nontrivial elements of G have prime order, all of the elements in a given cyclic subgroup fix the same vertices.*

Let H be isomorphic to S_4 and suppose that H acts faithfully on Γ. Then all elements of H of order 3 fix the same number of vertices, and all elements of H of order 4 fix the same number of vertices. All involutions of H which are in $G \cong A_4$ fix the same number of vertices, and all involutions of H which are not in G fix the same number of vertices.

We will also use the theorem below to focus on embeddings Γ of K_m in S^3 such that $\mathrm{TSG}_+(\Gamma)$ is induced by an isomorphic finite subgroup of $\mathrm{SO}(4)$ (the group of orientation preserving isometries of S^3). This theorem follows from a result in [6] together with the recently proved Geometrization Theorem [9].

Isometry Theorem *Let Ω be an embedding of some K_m in S^3. Then K_m can be re-embedded in S^3 as Γ such that $\mathrm{TSG}_+(\Omega) \le \mathrm{TSG}_+(\Gamma)$ and $\mathrm{TSG}_+(\Gamma)$ is induced by an isomorphic finite subgroup of $\mathrm{SO}(4)$.*

Suppose that Ω is an embedding of a complete graph K_m in S^3 such that $G = \mathrm{TSG}_+(\Omega)$ is isomorphic to A_4, A_5 or S_4. By applying the Isometry Theorem, we obtain a re-embedding Γ of K_m in S^3 such that $G \leq \mathrm{TSG}_+(\Gamma)$ is induced on Γ by an isomorphic finite subgroup $\widehat{G} \leq \mathrm{SO}(4)$. This simplifies our analysis since every finite order element of SO(4) is either a rotation with fixed point set a geodesic circle or a glide rotation with no fixed points. If the fixed point sets of two such rotations intersect but do not coincide, then they intersect in 2 points. Furthermore, if all of the elements of a finite subgroup of SO(4) pointwise fix the same simple closed curve, then that subgroup must be cyclic (this can be seen by looking at the action of the subgroup on the normal bundle).

For each $g \in G$, we let $\widehat{g}$ denote the element of $\widehat{G}$ which induces g. Since G has finite order, if $g \in G$ fixes both vertices of an edge, then $\widehat{g}$ pointwise fixes that edge. Since the fixed point set of every element of $\widehat{G}$ is either a circle or the empty set, no nontrivial element of G can fix more than 3 vertices of Γ. If $g \in G$ fixes 3 vertices, then $\mathrm{fix}(\widehat{g})$ is precisely these 3 fixed vertices together with the edges between them. Suppose that $g \in G$ fixes 3 vertices and has order 2. Then g must interchange some pair of vertices v and w in Γ. Thus $\widehat{g}$ must fix a point on the edge $\overline{vw}$. As this is not possible, no order 2 element of G fixes more than 2 vertices. Since $G \leq \mathrm{Aut}(K_m)$ and G is isomorphic to A_4, A_5 or S_4, $m \geq 4$. In particular, since no $g \in G$ fixes more than 3 vertices, each $g \in G$ is induced by precisely one $\widehat{g} \in \widehat{G}$. The following lemmas put further restrictions on the number of fixed vertices of each element of a given order.

Lemma 2.1 *Let $G \leq \mathrm{Aut}(K_m)$ which is isomorphic to A_4 or A_5. Suppose there is an embedding Γ of K_m in S^3 such that G is induced on Γ by an isomorphic subgroup $\widehat{G} \leq \mathrm{SO}(4)$. Then no order 2 element of G fixes more than 1 vertex of Γ.*

Proof As observed above, no order 2 elements of G fixes more than 2 vertices. Suppose some order 2 element of G fixes 2 vertices of Γ. Thus, by the Fixed Vertex Property, each order 2 element of $\widehat{G}$ fixes 2 vertices, and hence also pointwise fixes the edge between the 2 vertices. Now observe that two distinct involutions of $\widehat{G}$ cannot pointwise fix the same edge, since a cyclic group can have at most one element of order 2.

Observe that $\widehat{G}$ contains a subgroup $\widehat{H} \cong D_2$. Since D_2 contains 3 elements of order 2, Γ has 3 edges which are each pointwise fixed by precisely one order 2 element of $\widehat{G}$. We see as follows that each order 2 element of $\widehat{H}$ must setwise fix all 3 of these edges. Let $\widehat{g}$ and $\widehat{h}$ be order 2 elements of $\widehat{H}$, and let x and y be the vertices of the edge that is pointwise fixed by $\widehat{g}$. Since all elements of D_2 commute, $g(h(x)) = h(g(x)) = h(x)$, so $h(x)$ is fixed by g. Since x and y are the only vertices that are fixed by g,

$h(x) \in \{x, y\}$. Similarly for $h(y)$. So $\widehat{h}$ setwise fixes the edge $\overline{xy}$. It follows that each order 2 element of $\widehat{H}$ setwise fixes all 3 of these edges. This implies that each order 2 element fixes the midpoint of each of the 3 edges. These 3 midpoints determine a geodesic, which must be pointwise fixed by all 3 order 2 elements of $\widehat{H}$. But this is impossible since a cyclic group can have at most one element of order 2. □

Lemma 2.2 *Let $G \leq \mathrm{Aut}(K_m)$ which is isomorphic to A_4. Suppose there is an embedding Γ of K_m in S^3 such that G is induced on Γ by an isomorphic subgroup $\widehat{G} \leq \mathrm{SO}(4)$. If an order 2 element of G fixes some vertex v, then v is fixed by every element of G.*

Proof Suppose an order 2 element $\varphi_1 \in G$ fixes a vertex v. By Lemma 2.1, φ_1 fixes no other vertices of Γ. Since $G \cong A_4$, there is an involution $\varphi_2 \in G$ such that $\langle \varphi_1, \varphi_2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Now by the Orbits Lemma, φ_2 takes fixed vertices of φ_1 to fixed vertices of φ_1. Thus $\varphi_2(v) = v$. Hence v is fixed by $\langle \varphi_1, \varphi_2 \rangle$. Furthermore, all of the order 2 elements of G are in $\langle \varphi_1, \varphi_2 \rangle$. Thus v is the only vertex fixed by any order 2 element of G.

Let ψ be an order 3 element of G. Now $\psi\varphi_1\psi^{-1}$ has order 2 and fixes $\psi(v)$. Thus $\psi(v) = v$. Since $G = \langle \varphi_1, \varphi_2, \psi \rangle$, v is fixed by every element of G. □

Lemma 2.3 *Let $G \leq \mathrm{Aut}(K_m)$ which is isomorphic to A_4. Suppose there is an embedding Γ of K_m in S^3 such that G is induced on Γ by an isomorphic subgroup $\widehat{G} \leq \mathrm{SO}(4)$. If some order 2 element of G fixes a vertex of Γ, then no element of G fixes 3 vertices.*

Proof Suppose some order 2 element of G fixes a vertex v. By Lemma 2.2, every element of G fixes v. Suppose that G contains an element ψ which fixes 3 vertices. It follows from Lemma 2.1 that the order of ψ must be 3. Now let $g \in G$ have order 3 such that $\langle g, \psi \rangle$ is not cyclic. It follows from the Fixed Vertex property that $\mathrm{fix}(\widehat{\psi})$ and $\mathrm{fix}(\widehat{g})$ each consist of 3 vertices and 3 edges. Since $v \in \mathrm{fix}(\widehat{g}) \cap \mathrm{fix}(\widehat{\psi})$, there must be another point $x \in \mathrm{fix}(\widehat{g}) \cap \mathrm{fix}(\widehat{\psi})$. However, since two edges cannot intersect in their interiors, x must be a vertex of Γ. This implies that $\widehat{\psi}$ and $\widehat{g}$ pointwise fix the edge $\overline{xv}$. However, this is impossible since $\langle \psi, g \rangle$ is not cyclic. Thus no element of G fixes 3 vertices. □

Lemma 2.4 *Let $G \leq \mathrm{Aut}(K_m)$ which is isomorphic to A_5. Suppose there an embedding Γ of K_m in S^3 such that G is induced on Γ by an isomorphic subgroup $\widehat{G} \leq \mathrm{SO}(4)$. Then no element of G fixes 3 vertices.*

Proof Recall that the only even order elements of A_5 are involutions. By Lemma 2.1, no involution of G fixes more than 1 vertex. Let ψ be an element of G of odd order q and suppose that ψ fixes 3 vertices. Now G contains an involution φ such that $\langle\varphi,\psi\rangle \cong D_q$. Thus for every vertex x which is fixed by ψ, $\psi\varphi(x)=\varphi\psi^{-1}(x)=\varphi(x)$. Hence $\varphi(x)$ is also fixed by ψ. So φ setwise fixes the set of fixed vertices of ψ. Since ψ fixes 3 vertices and φ has order 2, φ must fix one of these 3 vertices v.

Let $H \leq G$ such that $H \cong A_4$ and H contains the involution φ. Then by Lemma 2.2, every element of H fixes v. Since φ fixes v and ψ fixes 3 vertices, it follows from Lemma 2.3 that $\psi \notin H$. Therefore $\langle\psi, H\rangle = G$, because A_5 has no proper subgroup containing A_4 as a proper subgroup. Hence every element of G fixes v. Now let $g \in G$ have order q such that $\langle g,\psi\rangle$ is not cyclic. By the Fixed Vertex Property, $\mathrm{fix}(\widehat{g})$ and $\mathrm{fix}(\widehat{\psi})$ each contain 3 vertices and 3 edges. Thus we can repeat the argument given in the proof of Lemma 2.3 to get a contradiction. □

Lemma 2.5 *Let $G \leq \mathrm{Aut}(K_m)$ which is isomorphic to A_5. Suppose there is an embedding Γ of K_m in S^3 such that G is induced on Γ by an isomorphic subgroup $\widehat{G} \leq \mathrm{SO}(4)$. If an element $\psi \in G$ with odd order q fixes precisely one vertex v, then v is fixed by every element of G and no other vertex is fixed by any nontrivial element of G.*

Proof There is an involution $\varphi \in G$ such that $\langle\varphi,\psi\rangle \cong D_q$. Now $\psi\varphi(v)=\varphi\psi^{-1}(v)=\varphi(v)$. Since v is the only vertex fixed by ψ, we must have $\varphi(v)=v$. Now G contains a subgroup $H \cong A_4$ containing φ. By Lemma 2.2, since φ fixes v every element of H fixes v. Since A_4 does not contain D_3 or D_5, $\psi \notin H$. Hence as in the proof of Lemma 2.4, $\langle\psi, H\rangle = G$. Thus every element of G fixes v. Every involution in G is an element of a subgroup isomorphic to A_4. Thus by Lemma 2.1, v is the only vertex which is fixed by any involution in G.

Let $\beta \in G$ be of order $p=3$ or 5. Suppose β fixes some vertex $w \neq v$. Thus all of the elements in $\langle\beta\rangle \cong \mathbb{Z}_p$ fix v and w. Let n denote the number of subgroups of G that are isomorphic to $\mathbb{Z}_p$. Thus $n=6$ or $n=10$ according to whether $p=5$ or $p=3$ respectively. By the Fixed Vertex Property, all of the subgroups isomorphic to $\mathbb{Z}_p$ also fix 2 vertices. If $g \in G$ fixes w, then g pointwise fixes the edge $\overline{vw}$ and hence $\langle g,\beta\rangle$ is cyclic. It follows that each of the n subgroups isomorphic to $\mathbb{Z}_p$ fixes a distinct vertex in addition to v. These n vertices together with v span a subgraph $\Lambda \subseteq \Gamma$ which is an embedding of K_{n+1} such that Λ is setwise invariant under $\widehat{G}$ and $\widehat{G}$ induces an isomorphic group action on Λ. However, $n+1=7$ or 11. Since $G \cong A_5$ contains a subgroup isomorphic to D_2, this contradicts the $\boldsymbol{D_2}$ Lemma. Thus v is the only vertex which is fixed by any order p element of G. □

The following general result may be well known. However, since we could not find a reference, we include an elementary proof here. Observe that in contrast with Lemma 2.6, if $\widehat{G}$ acts on S^3 as the orientation preserving isometries of a regular 4–simplex then the order 5 elements are glide rotations.

Lemma 2.6 *Suppose that $\widehat{G} \leq \mathrm{SO}(4)$ such that $\widehat{G} \cong A_5$ and every order 5 element of $\widehat{G}$ is a rotation of S^3. Then $\widehat{G}$ induces the group of rotations of a regular solid dodecahedron.*

Proof The group $\widehat{G}$ contains subgroups $J_1, \dots, J_6$ which are isomorphic to $\mathbb{Z}_5$ and involutions $\varphi_1, \dots, \varphi_6$ such that for each i, $H_i = \langle J_i, \varphi_i \rangle \cong D_5$. Now since every order 5 element of $\widehat{G}$ is a rotation of S^3, for each i there is a geodesic circle L_i which is pointwise fixed by every element of J_i. Furthermore, because $H_i \cong D_5$, the circle L_i must be inverted by the involution φ_i. Hence there are points p_i and q_i on L_i which are fixed by φ_i. Now every involution in $H_i \cong D_5$ is conjugate to φ_i by an element of J_i. Hence every involution in H_i also fixes both p_i and q_i. For each i, let S_i denote the geodesic sphere S_i which meets the circle L_i orthogonally in the points p_i and q_i. Now S_i is setwise invariant under every element of H_i.

By analyzing the structure of A_5, we see that each involution in H_1 is also contained in precisely one of the groups $H_2, \dots, H_6$. Thus for each $i \neq 1$, the 2 points which are fixed by the involution in $H_1 \cap H_i$ are contained in $S_1 \cap S_i$. Since p_1 is fixed by every involution in H_1, it follows that p_1 is contained in every S_i. Observe that the set of geodesic spheres $\{S_1, \dots, S_6\}$ is setwise invariant under $\widehat{G}$. Since p_1 is in every S_i, this implies that the orbit P of p_1 is contained in every S_i.

If p_1 is fixed by every element of $\widehat{G}$, then $\widehat{G}$ induces the group of rotations of a regular solid dodecahedron centered at p_1. Thus we assume that p_1 is not fixed by every element of $\widehat{G}$. Since $\widehat{G}$ can be generated by elements of order 5, it follows that some order 5 element of $\widehat{G}$ does not fix p_i. The orbit of p_1 under that element must contain at least 5 elements, and hence $|P| \geq 5$. Suppose that some $S_i \neq S_1$. Then $S_1 \cap S_i$ consists of a geodesic circle C containing the set P. Since $|P| > 2$, the circle C is uniquely determined by P.

Now C must be setwise invariant under $\widehat{G}$ since P is. Thus the core D of the open solid torus $S^3 - C$ is also setwise invariant under $\widehat{G}$. Since a pair of circles cannot be pointwise fixed by a nontrivial orientation preserving isometry of S^3, $\widehat{G}$ induces a faithful action of $C \cup D$ taking each circle to itself. But the only finite groups that can act faithfully on a circle are cyclic or dihedral, and A_5 is not the product of two such groups. Thus every $S_i = S_1$.

Recall that for each i, the geodesic circle L_i is orthogonal to the sphere S_i and is pointwise fixed by every element of J_i. Since all of the geodesic circles $L_1, \dots, L_6$

are orthogonal to the single sphere $S_i = S_1$, they must all meet at a point x in a ball bounded by S_1. Now $\widehat{G} = \langle J_1, J_2 \rangle$, and every element of J_1 and J_2 fixes x. Thus $\widehat{G}$ fixes the point x. Hence again $\widehat{G}$ induces the group of rotations of a solid dodecahedron centered at the point x. $\square$

Suppose that G is a group acting faithfully on K_m. Let V denote the vertices of K_m and let $|\mathrm{fix}(g|V)|$ denote the number of vertices of K_m which are fixed by $g \in G$. Burnside's Lemma [2] gives us the following equation:

$$\#\text{ vertex orbits} = \frac{1}{|G|} \sum_{g \in G} |\mathrm{fix}(g|V)|$$

We shall use the fact that the left side of this equation is an integer to prove the necessity of our conditions for K_m to have an embedding Γ such that $G = \mathrm{TSG}_+(\Gamma)$ is isomorphic to A_4 or A_5. By the Fixed Vertex Property, all elements of the same order fix the same number of vertices of Γ. So we will use n_k to denote the number of fixed vertices of an element of G of order k. Observe that n_1 is always equal to m.

Theorem 2.7 *If a complete graph K_m has an embedding Γ in S^3 with $\mathrm{TSG}_+(\Gamma) \cong A_4$, then $m \equiv 0, 1, 4, 5, 8 \pmod{12}$.*

Proof Let $G = \mathrm{TSG}_+(\Gamma) \cong A_4$. By applying the Isometry Theorem, we obtain a re-embedding Λ of K_m such that G is induced on Λ by an isomorphic subgroup $\widehat{G} \leq \mathrm{SO}(4)$. Thus we can apply our lemmas. Note that $|A_4| = 12$, and A_4 contains 3 order 2 elements and 8 order 3 elements. Thus Burnside's Lemma tells us that $\frac{1}{12}(m + 3n_2 + 8n_3)$ is an integer.

By Lemma 2.1, we know that $n_2 = 0$ or 1, and by Lemma 2.3 we know that if $n_2 = 1$ then $n_3 \neq 3$. Also, by Lemma 2.2, if $n_3 = 0$, then $n_2 = 0$. So there are 5 cases, summarized in the table below. In each case, the value of $m \pmod{12}$ is determined by knowing that $\frac{1}{12}(m + 3n_2 + 8n_3)$ is an integer.

n_2	n_3	$m \pmod{12}$
0	0 or 3	0
0	1	4
0	2	8
1	1	1
1	2	5

This completes the proof. $\square$

Theorem 2.8 *If a complete graph K_m has an embedding Γ in S^3 with $\mathrm{TSG}_+(\Gamma) \cong A_5$, then $m \equiv 0, 1, 5, 20 \pmod{60}$.*

Proof Let $G = \mathrm{TSG}_+(\Gamma) \cong A_5$. By applying the Isometry Theorem, we obtain a re-embedding Λ of K_m such that G is induced on Λ by an isomorphic subgroup $\widehat{G} \le \mathrm{SO}(4)$. Note that $|A_5| = 60$, and A_5 contains 15 elements of order 2, 20 elements of order 3, and 24 elements of order 5. Thus Burnside's Lemma tells us that $\frac{1}{60}(m + 15n_2 + 20n_3 + 24n_5)$ is an integer.

By Lemma 2.4, for every $k > 1$, $n_k < 3$. Every element of G of order 2 or 3 is contained in some subgroup isomorphic to A_4. Thus as in the proof of Theorem 2.7, we see that $n_2 = 0$ or 1, and if $n_3 = 0$ then $n_2 = 0$. Also, by Lemma 2.5, if either $n_3 = 1$ or $n_5 = 1$, then all of n_2, n_3 and n_5 are 1.

Suppose that $n_5 = 2$. Then each order 5 element of $\widehat{G}$ must be a rotation. Let $\widehat{\psi}$, $\widehat{\varphi} \in \widehat{G}$ such that $\widehat{\psi}$ has order 5, $\widehat{\varphi}$ has order 2, and $\langle \widehat{\psi}, \widehat{\varphi} \rangle \cong D_5$. Then there is a circle which is fixed pointwise by $\widehat{\psi}$ and inverted by $\widehat{\varphi}$. Thus $\mathrm{fix}(\widehat{\varphi})$ intersects $\mathrm{fix}(\widehat{\psi})$ in 2 precisely points. By Lemma 2.6, we know that $\widehat{G}$ induces the group of rotations on a solid dodecahedron. Hence the fixed point sets of all of the elements of $\widehat{G}$ meet in two points, which are the points $\mathrm{fix}(\widehat{\varphi}) \cap \mathrm{fix}(\widehat{\psi})$. Now since $n_5 = 2$, $\mathrm{fix}(\widehat{\psi})$ contains precisely 2 vertices, and hence an edge e. Thus e must be inverted by $\widehat{\varphi}$. It follows that the midpoint of e is one of the two fixed points of $\widehat{G}$. Since G is not a dihedral group we know that e is not setwise invariant under every element of $\widehat{G}$. Thus there are other edges in the orbit of e which intersect e in its midpoint. Since two edges cannot intersect in their interiors, we conclude that $n_5 \neq 2$.

There are four cases summarized in the table below.

n_2	n_3	n_5	m (mod 60)
0	0	0	0
0	2	0	20
1	1	1	1
1	2	0	5

This completes the proof. □

Theorem 2.9 *If a complete graph K_m has an embedding Γ in S^3 with $\mathrm{TSG}_+(\Gamma) \cong S_4$, then $m \equiv 0, 4, 8, 12, 20 \pmod{24}$.*

Proof Let $G = \mathrm{TSG}_+(\Gamma) \cong S_4$. By applying the Isometry Theorem, we obtain a re-embedding Λ of K_m such that G is induced on Λ by an isomorphic subgroup $\widehat{G} \le \mathrm{SO}(4)$. Suppose that some order 4 element $\widehat{g} \in \widehat{G}$ has nonempty fixed point set. Then $\mathrm{fix}(\widehat{g}) \cong S^1$. Thus $\mathrm{fix}(\widehat{g}^2) = \mathrm{fix}(\widehat{g})$. Let (v_1, v_2, v_3, v_4) be a 4–cycle of vertices under g. Then g^2 inverts the edges $\overline{v_1 v_3}$ and $\overline{v_2 v_4}$. Thus $\mathrm{fix}(\widehat{g}^2)$ intersects both $\overline{v_1 v_3}$

and $\overline{v_2v_4}$. Hence $\widehat{g}$ fixes a point on each of $\overline{v_1v_3}$ and $\overline{v_2v_4}$. But this is impossible since (v_1, v_2, v_3, v_4) is induced by g. Thus every order 4 element of $\widehat{G}$ has empty fixed point set. In particular, no order 4 element of G fixes any vertices of Γ. Thus $m \not\equiv 1 \pmod 4$. Since $A_4 \le S_4$, by Theorem 2.7, $m \equiv 0,\ 1 \pmod 4$. It follows that $m \equiv 0 \pmod 4$.

Suppose that $m = 24n + 16$ for some n. The group S_4 has 3 elements of order 2 which are contained in A_4, 6 elements of order 2 which are not contained in A_4, 8 elements of order 3, and 6 elements of order 4. By the Fixed Vertex Property, each of the elements of any one of these orders fixes the same number of vertices. So according to Burnside's Lemma, $\frac{1}{24}((24n + 16) + 3n_2 + 6n_2' + 8n_3 + 6n_4)$ is an integer, where n_2 denotes the number of fixed vertices of elements of order 2 which are contained in A_4 and n_2' denotes the number of fixed vertices of elements of order 2 which are not contained in A_4.

We saw above that $n_4 = 0$. By Lemma 2.1, $n_2 = 0$ or 1. However, since n_2 is the only term with an odd coefficient, we cannot have $n_2 = 1$. Also since the number of vertices $m = 24n+16 \equiv 1 \pmod 3$, each element of order 3 must fix one vertex. Thus $n_2 = 0$ and $n_3 = 1$. Hence $\frac{1}{24}(16 + 6n_2' + 8)$ is an integer. It follows that $n_2' = 0$, since $n_2' \not> 3$. Let ψ be an order 3 element of $\widehat{G}$. Since $n_3 = 1$, ψ must be a rotation about a circle L containing a single vertex v. Since $\widehat{G} \cong S_4$, there is an involution $\varphi \in \widehat{G}$ such that $\langle \psi, \varphi \rangle \cong D_3$. It follows that φ inverts L. However, since v is the only vertex on L, $\varphi(v) = v$. This is impossible since $n_2 = n_2' = 0$. Thus $m \not\equiv 16 \pmod{24}$. The result follows. □

3 Embedding lemmas

For a given n, we would like to be able to construct an embedding of K_m which has a particular topological symmetry group. We do this by first embedding the vertices of K_m so that they are setwise invariant under a particular group of isometries, and then we embed the edges of K_m using the results below. Note that Lemma 3.1 applies to any finite group G of diffeomorphisms of S^3, regardless of whether the diffeomorphisms in G are orientation reversing or preserving.

Lemma 3.1 *Let G be a finite group of diffeomorphisms of S^3 and let γ be a graph whose vertices are embedded in S^3 as a set V such that G induces a faithful action on γ. Let Y denote the union of the fixed point sets of all of the nontrivial elements of G. Suppose that adjacent pairs of vertices in V satisfy the following hypotheses:*

(1) *If a pair is pointwise fixed by nontrivial elements h, $g \in G$, then* $\mathrm{fix}(h) = \mathrm{fix}(g)$.

(2) *No pair is interchanged by an element of G.*

(3) *Any pair that is pointwise fixed by a nontrivial $g \in G$ bounds an arc in* $\mathrm{fix}(g)$ *whose interior is disjoint from* $V \cup (Y - \mathrm{fix}(g))$.

(4) *Every pair is contained in a single component of $S^3 - Y$.*

Then there is an embedding of the edges of γ such that the resulting embedding of γ is setwise invariant under G.

Proof We partition the edges of γ into sets F_1 and F_2, where F_1 consists of all edges of γ both of whose embedded vertices are fixed by some nontrivial element of G, and F_2 consists of the remaining edges of γ. Thus, each F_i is setwise invariant under G.

We first embed the edges in F_1 as follows. Let $\{f_1, \dots, f_m\}$ be a set of edges consisting of one representative from the orbit of each edge in F_1. Thus for each i, some nontrivial $g_i \in G$ fixes the embedded vertices of f_i. Furthermore, by hypothesis (1), $\mathrm{fix}(g_i)$ is uniquely determined by f_i. By hypothesis (3), the vertices of f_i bound an arc $A_i \subseteq \mathrm{fix}(g_i)$ whose interior is disjoint from V and from the fixed point set of any element of G whose fixed point set is not $\mathrm{fix}(g_i)$. We embed the edge f_i as the arc A_i. Now it follows from our choice of A_i that the interiors of the arcs in the orbits of $A_1, \dots, A_m$ are pairwise disjoint.

Now let f be an edge in $F_1 - \{f_1, \dots, f_m\}$. Then for some $g \in G$ and some edge f_i, we have $g(f_i) = f$. We embed the edge f as $g(A_i)$. To see that this is well-defined, suppose that for some $h \in G$ and some f_j, we also have $f = h(f_j)$. Then $i = j$, since we picked only one representative from each edge orbit. Therefore $g(f_i) = h(f_i)$. This implies $h^{-1}g$ fixes both vertices of f_i since by hypothesis (2) no edge of γ is inverted by G. Now, by hypothesis (1), if $h^{-1}g$ is nontrivial, then $\mathrm{fix}(h^{-1}g) = \mathrm{fix}(g_i)$. Since $A_i \subseteq \mathrm{fix}(g_i)$, it follows that $h(A_i) = g(A_i)$, as desired. We can thus unambiguously embed all of the edges of F_1. Let E_1 denote this set of embedded edges. By our construction, E_1 is setwise invariant under G.

Next we will embed the edges of F_2. Let $\pi\colon S^3 \to S^3/G$ denote the quotient map. Then $\pi|(S^3 - Y)$ is a covering map, and the quotient space $Q = (S^3 - Y)/G$ is a 3–manifold. We will embed representatives of the edges of F_2 in the quotient space Q, and then lift them to get an embedding of the edges in S^3.

Let $\{e_1, \dots, e_n\}$ be a set of edges consisting of one representative from the orbit of each edge in F_2. For each i, let x_i and y_i be the embedded vertices of e_i in V. By hypothesis (4), for each $i = 1, \dots, n$, there exists a path α_i in S^3 from x_i to y_i

whose interior is disjoint from $V \cup Y$. Let $\alpha_i' = \pi \circ \alpha_i$. Then α_i' is a path or loop from $\pi(x_i)$ to $\pi(y_i)$ whose interior is in Q. Using general position in Q, we can homotop each α_i', fixing its endpoints, to a simple path or loop ρ_i' such that the interiors of the $\rho_i'(I)$ are pairwise disjoint and are each disjoint from $\pi(V \cup Y)$. Now, for each i, we lift ρ_i' to a path ρ_i beginning at x_i. Then each $\mathrm{int}(\rho_i)$ is disjoint from $V \cup Y$. Since $\rho_i' = \pi \circ \rho_i$ is one-to-one except possibly on the set $\{0, 1\}$, ρ_i must also be one-to-one except possibly on the set $\{0, 1\}$. Also, since ρ_i' is homotopic fixing its endpoints to α_i', ρ_i is homotopic fixing its endpoints to α_i. In particular, ρ_i is a simple path from x_i to y_i. We embed the edge e_i as $\rho_i(I)$.

Next, we will embed an arbitrary edge e of F_2. By hypothesis (2) and the definition of F_2, no edge in F_2 is setwise invariant under any nontrivial element of G. Hence there is a unique $g \in G$ and a unique $i \leq n$ such that $e = g'(e_i)$. It follows that e determines a unique arc $g(\rho_i(I))$ between $g(x_i)$ and $g(y_i)$. We embed e as $g(\rho_i(I))$. By the uniqueness of g and i, this embedding is well-defined. Let E_2 denote the set of embedded edges of F_2. Then G leaves E_2 setwise invariant.

Now, since each $\mathrm{int}(\rho_i'(I))$ is disjoint from $\pi(V)$, the interior of each embedded edge of E_2 is disjoint from V. Similarly, since $\rho_i'(I)$ and $\rho_j'(I)$ have disjoint interiors when $i \neq j$, for every g, $h \in G$, $g(\rho_i(I))$ and $h(\rho_j(I))$ also have disjoint interiors when $i \neq j$. And since ρ_i' is a simple path or loop whose interior is disjoint from $\pi(Y)$, if $g \neq h$, then $g(\rho_i(I))$ and $h(\rho_i(I))$ have disjoint interiors. Thus the embedded edges of E_2 have pairwise disjoint interiors.

Let Γ consist of the set of embedded vertices V together with the set of embedded edges $E_1 \cup E_2$. Then Γ is setwise invariant under G. Also, every edge in E_1 is an arc of Y, whose interior is disjoint from V, and the interior of every edge in E_2 is a subset of $S^3 - (Y \cup V)$. Therefore the interiors of the edges in E_1 and E_2 are disjoint. Hence Γ is an embedded graph with underlying abstract graph γ, and Γ is setwise invariant under G. □

We use Lemma 3.1 to prove the following result. Note that $\mathrm{Diff}_+(S^3)$ denotes the group of orientation preserving diffeomorphisms of S^3. Thus by contrast with Lemma 3.1, the Edge Embedding Lemma only applies to finite groups of orientation preserving diffeomorphisms of S^3.

Edge Embedding Lemma *Let G be a finite subgroup of $\mathrm{Diff}_+(S^3)$ and let γ be a graph whose vertices are embedded in S^3 as a set V such that G induces a faithful action on γ. Suppose that adjacent pairs of vertices in V satisfy the following hypotheses:*

(1) *If a pair is pointwise fixed by nontrivial elements h, $g \in G$, then $\mathrm{fix}(h) = \mathrm{fix}(g)$.*

(2) *For each pair* $\{v, w\}$ *in the fixed point set* C *of some nontrivial element of* G*, there is an arc* $A_{vw} \subseteq C$ *bounded by* $\{v, w\}$ *whose interior is disjoint from* V *and from any other such arc* $A_{v'w'}$*.*

(3) *If a point in the interior of some* A_{vw} *or a pair* $\{v, w\}$ *bounding some* A_{vw} *is setwise invariant under an* $f \in G$*, then* $f(A_{vw}) = A_{vw}$*.*

(4) *If a pair is interchanged by some* $g \in G$*, then the subgraph of* γ *whose vertices are pointwise fixed by* g *can be embedded in a proper subset of a circle.*

(5) *If a pair is interchanged by some* $g \in G$*, then* $\mathrm{fix}(g)$ *is nonempty, and for any* $h \neq g$*, then* $\mathrm{fix}(h) \neq \mathrm{fix}(g)$*.*

Then there is an embedding of the edges of γ *in* S^3 *such that the resulting embedding of* γ *is setwise invariant under* G*.*

Proof Let γ' denote γ together with a valence 2 vertex added to the interior of every edge whose vertices are interchanged by some element of G. Then G induces a faithful action on γ', since G induces a faithful action on γ. For each $g \in G$ we will let g' denote the automorphism of γ' induced by g, and let G' denote the group of automorphisms of γ' induced by G. No element of G' interchanges a pair of adjacent vertices of γ'. Since G induces a faithful action on γ', each $g' \in G$ is induced by a unique $g \in G$.

Let M denote the set of vertices of γ' which are not in γ. Each vertex $m \in M$ is fixed by an element $f' \in G'$ which interchanges the pair of vertices adjacent to m. We partition M into sets M_1 and M_2, where M_1 contains those vertices of M whose adjacent vertices are both fixed by a nontrivial automorphism in G' and M_2 contains those vertices of M whose adjacent vertices are not both fixed by a nontrivial automorphism in G'.

We first embed the vertices of M_1. Let $\{m_1, \dots, m_r\}$ be a set consisting of one representative from the orbit of each vertex in M_1, and for each m_i, let v_i and w_i denote the vertices which are adjacent to the vertex m_i in γ'. Thus v_i and w_i are adjacent vertices of γ. By definition of M_1, each m_i is fixed both by some automorphism $f_i' \in G'$ which interchanges v_i and w_i and by some element $h_i' \in G'$ which fixes both v_i and w_i. Let f_i and h_i be the elements of G which induce f_i' and h_i' respectively. Let $A_{v_i w_i}$ denote the arc in $\mathrm{fix}(h_i)$ given by hypothesis (2). Since f_i interchanges v_i and w_i, it follows from hypothesis (3) that $f_i(A_{v_i w_i}) = A_{v_i w_i}$. Also since f_i has finite order, there is a unique point x_i in the interior of $A_{v_i w_i}$ which is fixed by f_i. We embed m_i as the point x_i. By hypothesis (2), if $i \neq j$, then $A_{v_i w_i}$ and $A_{v_j w_j}$ have disjoint interiors, and hence $x_i \neq x_j$.

We see as follows that the choice of x_i does not depend on the choice of either h_i or f_i. Suppose that m_i is fixed by some $f' \in G'$ which interchanges v_i and w_i and some $h' \in G'$ which fixes both v_i and w_i. Let f and h be the elements of G which induce f' and h' respectively. Since both f and h leave the pair $\{v_i, w_i\}$ setwise invariant, by hypothesis (3) both f and h leave the arc $A_{v_i w_i}$ setwise invariant. Since h has finite order and fixes both v_i and w_i, h pointwise fixes the arc $A_{v_i w_i}$, and hence $\mathrm{fix}(h) = \mathrm{fix}(h_i)$. Thus the choice of the arc $A_{v_i w_i}$ does not depend on h. Also since G has finite order, and f and f_i both interchange v_i and w_i leaving $A_{v_i w_i}$ setwise invariant, $f^{-1} f_i$ pointwise fixes $A_{v_i w_i}$. Hence $f | A_{v_i w_i} = f_i | A_{v_i w_i}$, and thus the choice of x_i is indeed independent of f_i and h_i. In fact, by the same argument we see that x_i is the unique point in the interior of $A_{v_i w_i}$ that is fixed by an element of G which setwise but not pointwise fixes $A_{v_i w_i}$ (we will repeatedly use this fact below).

Now let m denote an arbitrary point in M_1. Then for some i and some automorphism $g' \in G'$, $m = g'(m_i)$. Let v and w be the vertices that are adjacent to the vertex m in γ'. Then $\{v, w\} = g'(\{v_i, w_i\})$. Let g be the element of G which induces g'. We embed m as the point $g(x_i)$. To see that this embedding is unambiguous, suppose that for some other automorphism $\varphi' \in G'$, we also have $m = \varphi'(m_j)$. Then $j = i$, since the orbits of $m_1, \dots, m_r$ are disjoint. Let φ denote the element of G which induces φ'. Then $g^{-1}\varphi(m_i) = m_i$, and hence $g^{-1}\varphi(\{v_i, w_i\}) = \{v_i, w_i\}$. It follows from hypothesis (3) that $g^{-1}\varphi(A_{v_i w_i}) = A_{v_i w_i}$. Now x_i is the unique point in the interior of $A_{v_i w_i}$ that is fixed by an element of G which setwise but not pointwise fixes $A_{v_i w_i}$. It follows that $\varphi(x_i) = g(x_i)$. Thus our embedding is well defined for all of the points of M_1. Furthermore, since $m_1, \dots, m_r$ have distinct orbits under G', the pairs $\{v_1, w_1\}, \dots, \{v_r, w_r\}$ have distinct orbits under G. Hence the arcs $A_{v_1 w_1}, \dots, A_{v_r w_r}$ are not only disjoint, they also have distinct orbits under G. Thus the points of M_1 are embedded as distinct points of S^3.

Next we embed the vertices of M_2. Let $\{g'_1, \dots, g'_q\}$ consist of one representative from each conjugacy class of automorphisms in G' which fix a point in M_2, and let each g'_i be induced by $g_i \in G$. For each g'_i, from the set of vertices of M_2 that are fixed by that g'_i, choose a subset $\{p_{i1}, \dots, p_{in_i}\}$ consisting of one representative from each of their orbits under G.

Let $F_i = \mathrm{fix}(g_i)$. By hypothesis (5) and Smith Theory [11], $F_i \cong S^1$ and F_i is not the fixed point set of any element of G other than g_i. Thus each arc A_{vw} that is a subset of F_i corresponds to some edge of γ whose vertices are fixed by g_i. By hypothesis (4) the subgraph of γ whose vertices are fixed by g_i is homeomorphic to a proper subset of a circle. Furthermore, since $G \leq \mathrm{Diff}_+(S^3)$ is finite, the fixed point set of any nontrivial element of G other than g_i meets F_i in either 0 or 2 points. Thus we can choose an arc $A_i \subseteq F_i$ which does not intersect any A_{vw}, is disjoint from the fixed point set of any other nontrivial element of G, and is disjoint from its own image under any other nontrivial element of G. Now we can choose a set $\{y_{i1}, \dots, y_{in_i}\}$ of distinct points in the arc A_i, and embed the set of vertices $\{p_{i1}, \dots, p_{in_i}\}$ as the set $\{y_{i1}, \dots, y_{in_i}\}$. Observe that if some p_{ij} were also fixed by a nontrivial automorphism $g' \in G'$ such that $g' \neq g'_i$, then either g' or $g'g'_i$ would fix both vertices adjacent to p_{ij}, which is contrary to the definition of M_2. Hence g'_i is the unique nontrivial automorphism in G' fixing p_{ij}. Thus our embedding of p_{ij} is well defined.

We embed an arbitrary point p of M_2 as follows. Choose i, j, and $g' \in G'$ such that $p = g'(p_{ij})$, and g' is induced by a unique element $g \in G$. Since p_{ij} is embedded as a point $y_{ij} \in A_i \subseteq \mathrm{fix}(g_i)$, we embed p as $g(y_{ij})$. To see that this is well defined, suppose that for some automorphism $\varphi' \in G'$ we also have $p = \varphi'(p_{lk})$, and φ' is induced by $\varphi \in G$. Then $p_{ij} = p_{lk}$, since their orbits share the point p, and hence are equal. Now $(g')^{-1}\varphi'(p_{ij}) = p_{ij}$. But since $p_{ij} \in M_2$, no nontrivial element of G' other than h'_i fixes p_{ij}. Thus either $(g')^{-1}\varphi' = g'_i$ or $g' = \varphi'$. In the former case, the diffeomorphism $g^{-1}\varphi = g_i$ fixes the point y_{ij}, since $y_{ij} \in \mathrm{fix}(g_i)$. Hence in either case $g(y_{ij}) = \varphi(y_{ij})$. Thus our embedding is well defined for all points of M_2.

Recall that, if $i \neq j$, then g_i and g_j are in distinct conjugacy classes of G. Also, by hypothesis (5), F_i is not fixed by any nontrivial element of G other than g_i. Now it follows that F_i is not in the orbit of F_j, and hence the points of M_2 are embedded as distinct points of S^3. Finally, since the points of M_1 are each embedded in an arc A_{vw} and the points of M_2 are each embedded in an arc which is disjoint from any A_{vw}, the sets of vertices in M_1 and M_2 have disjoint embeddings.

Let V' denote V together with the embeddings of the points of M. Thus we have embedded the vertices of γ' in S^3. We check as follows that the hypotheses of Lemma 3.1 are satisfied for V'. When we refer to a hypothesis of Lemma 3.1 we shall put an * after the number of the hypothesis to distinguish it from a hypothesis of the lemma we are proving. We have defined V' so that it is setwise invariant under G and G induces a faithful action on γ'. Also, by the definition of γ', hypothesis (2*) is satisfied. Since G is a finite subgroup of $\mathrm{Diff}_+(S^3)$, the union Y of all of the fixed point sets of the nontrivial elements of G is a graph in S^3. Thus $S^3 - Y$ is connected, and hence hypothesis (4*) is satisfied.

To see that hypothesis (1*) is satisfied for V', suppose a pair $\{x, y\}$ of adjacent vertices of γ' are both fixed by nontrivial elements $h, g \in G$. If the pair is in V, then they are adjacent in γ, and hence by hypothesis (1) we know that $\mathrm{fix}(h) = \mathrm{fix}(g)$. Thus suppose that $x \in M$. Then $x \in M_1$, since the vertices in M_2 are fixed by at most one nontrivial automorphism in G'. Now without loss of generality, we can assume that x is one of the $m_i \in M_1$ and y is an adjacent vertex v_i. Thus x is embedded as $x_i \in \mathrm{int}(A_{v_i w_i})$. Since h and g both fix x_i, by hypothesis (3), both $h(A_{v_i w_i}) = A_{v_i w_i}$ and $g(A_{v_i w_i}) = A_{v_i w_i}$. It follows that h and g both fix w_i, since we know they fix v_i. Now $\{v_i, w_i\}$ are an adjacent pair in γ. Hence again by hypothesis (1), $\mathrm{fix}(h) = \mathrm{fix}(g)$. It follows that hypothesis (1*) is satisfied for V'.

It remains to check that hypothesis (3*) is satisfied for V'. Let s and t be adjacent vertices of γ' which are fixed by some nontrivial $g \in G$. We will show that s and t bound an arc in $\mathrm{fix}(g)$ whose interior is disjoint from V', and if any $f \in G$ fixes a point in the interior of this arc then $\mathrm{fix}(f) = \mathrm{fix}(g)$. First suppose that s and t are both in V. Then no element of G interchanges s and t. Now, by hypothesis (2), s and t bound an arc $A_{st} \subseteq \mathrm{fix}(g)$ whose interior is disjoint from V. Furthermore, by (2), $\mathrm{int}(A_{st})$ is disjoint from any other A_{vw}, if $\{v, w\} \neq \{s, t\}$. Thus $\mathrm{int}(A_{st})$ is disjoint from the embedded points of M_1. Also the embedded points of M_2 are disjoint from any A_{vw}. Thus $\mathrm{int}(A_{st})$ is disjoint from V'. Suppose some $f \in G$ fixes a point in $\mathrm{int}(A_{st})$. Then by hypothesis (3), $f(A_{st}) = A_{st}$. So f either fixes or interchanges s and t. In the latter case, s and t would not be adjacent in γ'. Thus f fixes both s and t. Now by hypothesis (1), we must have $\mathrm{fix}(f) = \mathrm{fix}(g)$. So the pair of vertices s and t satisfy hypothesis (3*).

Next suppose that $s \in V' - V$. Since s and t are adjacent in γ', we must have $t \in V$. Now s is the embedding of some $m \in M$, and m is adjacent to vertices $v, t \in V$ which are both fixed by g. Thus $m \in M_1$. By our embedding of M_1, for some $h \in G$ and some i, we have $s = h(x_i)$ where x_i is adjacent to vertices v_i and w_i in γ'. It follows that $\{v, t\} = h(\{v_i, w_i\})$. Thus $h^{-1}gh$ fixes v_i and w_i. Let h_i and $A_{v_i w_i} \subseteq \mathrm{fix}(h_i)$ be as in the description of our embedding of the points in M_1. Thus v_i and w_i are adjacent vertices of γ which are fixed by both h_i and $h^{-1}gh$. By hypothesis (1), $\mathrm{fix}(h_i) = \mathrm{fix}(h^{-1}gh)$. Thus $A_{v_i w_i} \subseteq \mathrm{fix}(h^{-1}gh)$. Let $A = h(A_{v_i w_i})$. Then A is an arc bounded by v and t which is contained in $\mathrm{fix}(g)$. Furthermore, the interior of $A_{v_i w_i}$ is disjoint from V and x_i is the unique point in the interior of $A_{v_i w_i}$ that is fixed by an element of G which setwise but not pointwise fixes $A_{v_i w_i}$. Thus the interior of A is disjoint from V and $s = h(x_i)$ is the unique point in the interior of A that is fixed by an element of G which setwise but not pointwise fixes A. Let A_{st} denote the subarc of A with endpoints s and t. Then $\mathrm{int}(A_{st})$ is disjoint from V', and A_{st} satisfies hypothesis (3*).

Thus we can apply Lemma 3.1 to the embedded vertices of γ' to get an embedding of the edges of γ' such that the resulting embedding of γ' is setwise invariant under G. Now by omitting the vertices of $\gamma' - \gamma$ we obtain the required embedding of γ. $\square$

4 Embeddings with $\mathrm{TSG}_+(\Gamma) \cong S_4$

Recall from Theorem 2.9 that if K_m has an embedding Γ with $\mathrm{TSG}_+(\Gamma) \cong S_4$, then $m \equiv 0, 4, 8, 12, 20 \pmod{24}$. For each of these values of m, we will use the Edge Embedding Lemma to construct an embedding of K_m whose topological symmetry group is isomorphic to S_4.

Proposition 4.1 *Let* $m \equiv 0, 4, 8, 12, 20 \pmod{24}$. *Then there is an embedding* Γ *of* K_m *in* S^3 *such that* $\mathrm{TSG}_+(\Gamma) \cong S_4$.

Proof Let $G \cong S_4$ be the finite group of orientation preserving isometries of S^3 which leaves the 1–skeleton τ of a tetrahedron setwise invariant. Observe that every nontrivial element of G with nonempty fixed point set is conjugate to one of the rotations f, h, g or g^2 illustrated in Figure 1. Furthermore, an even order element of G has nonempty fixed point set if and only if it is an involution. Also, for every $g \in G$ of order 4, every point in S^3 has an orbit of size 4 under g, so g does not interchange any pair of vertices.Thus regardless of how we embed our vertices, hypothesis (5) of the Edge Embedding Lemma will be satisfied.

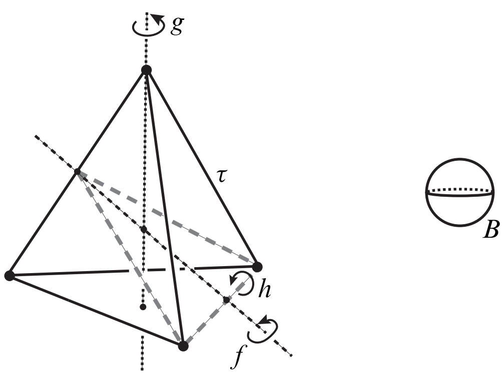

Figure 1: The isometry group G leaves the 1–skeleton τ of a tetrahedron setwise invariant. The ball B is disjoint from the fixed point set of any nontrivial element of G and from its image under every nontrivial element of G.

Let $n \geq 0$. We begin by defining an embedding of K_{24n}. Let B denote a ball which is disjoint from the fixed point set of any nontrivial element of G, and which is disjoint from its image under every nontrivial element of G. Choose n points in B, and let V_0 denote the orbit of these points under G. Since $|S_4| = 24$, the set V_0 contains $24n$ points. These points will be the embedded vertices of K_{24n}. Since none of the points in V_0 is fixed by any nontrivial element of G, it is easy to check that hypotheses (1)–(4) of the Edge Embedding Lemma are satisfied for the set V_0. Thus the Edge Embedding Lemma gives us an embedding Γ_0 of K_{24n} which is setwise invariant under G. It follows that $\mathrm{TSG}_+(\Gamma_0)$ contains a subgroup isomorphic to S_4. However, we know by the Complete Graph Theorem that S_4 cannot be isomorphic to a proper subgroup of $\mathrm{TSG}_+(\Gamma_0)$. Thus $S_4 \cong \mathrm{TSG}_+(\Gamma_0)$.

Next we will embed K_{24n+4}. Let V_4 denote the four corners of the tetrahedron τ (illustrated in Figure 1). We embed the vertices of K_{24n+4} as the points in $V_4 \cup V_0$. Now the edges of τ are the arcs required by hypothesis (2) of the Edge Embedding Lemma. Thus it is not hard to check that the set $V_4 \cup V_0$ satisfies the hypotheses of the Edge Embedding Lemma. By applying the Edge Embedding Lemma and the Complete Graph Theorem as above we obtain an embedding Γ_4 of K_{24n+4} such that $S_4 \cong \mathrm{TSG}_+(\Gamma_4)$.

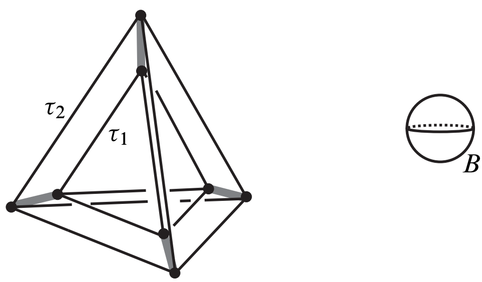

Figure 2: The points of V_8 are the vertices of $\tau_1 \cup \tau_2$. The arcs required by hypothesis (2) are the gray arcs between corresponding vertices.

Next we will embed K_{24n+8}. Let T denote a regular solid tetrahedron with 1–skeleton τ. Let τ_1 denote the 1–skeleton of a tetrahedron contained in T and let τ_2 denote the 1–skeleton of a tetrahedron in $S^3 - T$ such that $\tau_1 \cup \tau_2$ is setwise invariant under G. Observe that τ_1 and τ_2 are interchanged by all elements conjugate to h in Figure 1, and each τ_i is setwise fixed by all the other elements of G. We obtain the graph illustrated in Figure 2 by connecting τ_1 and τ_2 with arcs contained in the fixed point sets of the elements of G of order 3. Now let V_8 denote the vertices of $\tau_1 \cup \tau_2$. Then V_8 is setwise invariant under G. We embed the vertices of K_{24n+8} as the points of $V_8 \cup V_0$. It is easy to check that hypothesis (1) of the Edge Embedding Lemma is satisfied. To check hypothesis (2), first observe that the only pairs of vertices that are fixed by a nontrivial element of G are the pairs of endpoints of the arcs joining τ_1 and τ_2 (illustrated as gray arcs in Figure 2). These arcs are precisely those required by hypothesis (2). Now hypotheses (3) and (4) follow easily. Thus again by applying the

Edge Embedding Lemma and the Complete Graph Theorem we obtain an embedding Γ_8 of K_{24n+8} such that $S_4 \cong \mathrm{TSG}_+(\Gamma_8)$.

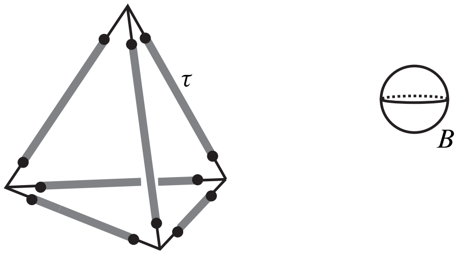

Figure 3: The points of V_{12} are symmetrically placed on the edges of τ. The arcs required by hypothesis (2) are the gray arcs between vertices on the same edge of τ.

Next we will embed K_{24n+12}. Let V_{12} be a set of 12 vertices which are symmetrically placed on the edges of the tetrahedron τ so that V_{12} is setwise invariant under G (see Figure 3). We embed the vertices of K_{24n+12} as $V_{12} \cup V_0$. It is again easy to check that hypothesis (1) of the Edge Embedding Lemma is satisfied by $V_{12} \cup V_0$. To check hypothesis (2) observe that the only pairs of vertices that are fixed by a nontrivial element of G are pairs on the same edge of τ. The arcs required by hypothesis (2) in Figure 3 are illustrated as gray arcs. Hypotheses (3) and (4) now follow easily. Thus again by applying the Edge Embedding Lemma and the Complete Graph Theorem we obtain an embedding Γ_{12} of K_{24n+12} such that $S_4 \cong \mathrm{TSG}_+(\Gamma_{12})$.

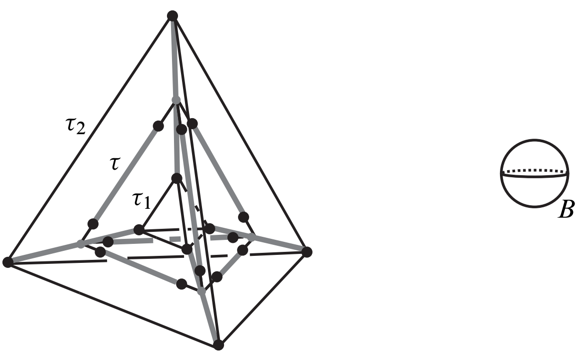

Figure 4: The points of $V_8 \cup V_{12}$ are the vertices of $\tau \cup \tau_1 \cup \tau_2$. The gray arcs required by hypothesis (2) are the union of those in Figures 2 and 3.

Finally, in order to embed K_{24n+20} we first embed the vertices as $V_0 \cup V_8 \cup V_{12}$ from Figure 2 and Figure 3. In Figure 4, the 20 vertices of $V_8 \cup V_{12}$ are indicated by black dots and the arcs required by hypothesis (2) are highlighted in gray. These vertices and arcs are the union of those illustrated in Figures 2 and 3. Now again by applying the Edge Embedding Lemma and the Complete Graph Theorem we obtain an embedding Γ_{12} of K_{24n+8} such that $S_4 \cong \mathrm{TSG}_+(\Gamma_{12})$. □

The following theorem summarizes our results on when a complete graph can have an embedding whose topological symmetry group is isomorphic to S_4.

S_4 Theorem *A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that $\mathrm{TSG}_+(\Gamma) \cong S_4$ if and only if $m \equiv 0, 4, 8, 12, 20 \pmod{24}$.*

5 Embeddings Γ with $\mathrm{TSG}_+(\Gamma) \cong A_5$

Recall from Theorem 2.8 that if K_m has an embedding Γ in S^3 such that $G = \mathrm{TSG}_+(\Gamma) \cong A_5$, then $m \equiv 0, 1, 5 \text{ or } 20 \pmod{60}$. In this section we show that for all of these values of m there is an embedding of K_m whose topological symmetry group is isomorphic to A_5.

Proposition 5.1 *Let $m \equiv 0, 1, 5, 20 \pmod{60}$. Then there exists an embedding Γ of K_m in S^3 such that $\mathrm{TSG}_+(\Gamma) \cong A_5$.*

Proof Let $G \cong A_5$ denote the finite group of orientation preserving isometries of S^3 which leaves a regular solid dodecahedron D setwise invariant. Every element of this group is a rotation, and hence has nonempty fixed point set. Also the only even order elements of A_5 are involutions. Thus regardless of how we embed our vertices, hypothesis (5) of the Edge Embedding Lemma will be satisfied for the group G. Let $H \cong A_5$ denote the finite group of orientation preserving isometries of S^3 which leaves a regular 4–simplex σ setwise invariant. Observe that the elements of order 2 of H interchange pairs of vertices of the 4–simplex and hence have nonempty fixed point sets. Thus regardless of how we embed our vertices, hypothesis (5) of the Edge Embedding Lemma will be satisfied for the group H. We will use either G or H for each of our embeddings.

We shall use G to embed K_{60n}. Let B be a ball which is disjoint from the fixed point set of any nontrivial element of G and which is disjoint from its image under every nontrivial element of G. Choose n points in B, and let V_0 denote the orbit of these points under G. We embed the vertices of K_{60n} as the points of V_0. Since none of the points of V_0 is fixed by any nontrivial element of G, the hypotheses of the Edge Embedding Lemma are easy to check. Thus by applying the Edge Embedding Lemma and the Complete Graph Theorem, we obtain an embedding Γ_0 of K_{60n} in S^3 such that $A_5 \cong \mathrm{TSG}_+(\Gamma_0)$.

In order to embed K_{60n+1} we again use the isometry group G. We embed the vertices of K_{60n+1} as $V_0 \cup \{x\}$, where x is the center of the invariant solid dodecahedron D. Since x is the only vertex which is fixed by a nontrivial element of G, the hypotheses of the Edge Embedding Lemma are satisfied for $V_0 \cup \{x\}$. Thus as above, by applying the Edge Embedding Lemma and the Complete Graph Theorem, we obtain an embedding Γ_1 of K_{60n+1} in S^3 such that $A_5 \cong \mathrm{TSG}_+(\Gamma_1)$.

In order to embed K_{60n+5} we use the isometry group H. Let B' be a ball which is disjoint from the fixed point set of any nontrivial element of H and which is disjoint from its image under every nontrivial element of H. Thus B' is disjoint from the 4–simplex σ. Choose n points in B', and let W_0 denote the orbit of these points under H. Let W_5 denote the set of vertices of the 4–simplex σ. We embed the vertices of K_{60n+5} as the points of $W_0 \cup W_5$. Now $W_0 \cup W_5$ is setwise invariant under H, and H induces a faithful action of K_{60n+5}. The arcs required by hypothesis (2) of the Edge Embedding Lemma are the edges of the 4–simplex σ. Thus it is easy to check that the hypotheses of the Edge Embedding Lemma are satisfied for $W_0 \cup W_5$. Hence as above by applying the Edge Embedding Lemma and the Complete Graph Theorem, we obtain an embedding Γ_5 of K_{60n+5} in S^3 such that $A_5 \cong \mathrm{TSG}_+(\Gamma_5)$.

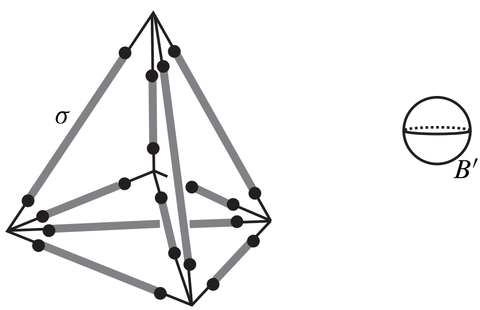

Figure 5: The points of W_{20} are symmetrically placed on the 4–simplex σ. The arcs required by hypothesis (3) are the gray arcs between vertices on the same edge of σ.

Finally, in order to embed K_{60n+20} we again use the isometry group H. Observe that each order 2 element of H fixes one vertex of σ, each order 3 element of H fixes 2 vertices of σ, and each order 5 element of H fixes no vertices of σ. Let W_{20} denote a set of 20 points which are symmetrically placed on the edges of the 4–simplex σ so that W_{20} is setwise invariant under H (see Figure 5). We embed the vertices of K_{60n+20} as the points of $W_0 \cup W_{20}$. The only pairs of points in W_{20} that are both fixed by a single nontrivial element of H are on the same edge of the 4–simplex σ and are fixed by two elements of order 3. We illustrate the arcs required by hypothesis (2) of the Edge Embedding Lemma as gray arcs in Figure 5. Now as above, by applying the Edge Embedding Lemma and the Complete Graph Theorem, we obtain an embedding Γ_{20} of K_{60n+20} in S^3 such that $A_5 \cong \mathrm{TSG}_+(\Gamma_{20})$. □

The following theorem summarizes our results on when a complete graph can have an embedding whose topological symmetry group is isomorphic to A_5.

A_5 Theorem *A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that $\mathrm{TSG}_+(\Gamma) \cong A_5$ if and only if $m \equiv 0, 1, 5, 20 \pmod{60}$.*

6 Embeddings Γ with $\mathrm{TSG}_+(\Gamma) \cong A_4$

Recall from Theorem 2.7 that if K_m has an embedding Γ in S^3 such that $G = \mathrm{TSG}_+(\Gamma) \cong A_4$, then $m \equiv 0, 1, 4, 5$ or $8 \pmod{12}$. In this section we will show that for these values of m, there are embeddings of K_m whose topological symmetry group is isomorphic to A_4.

We begin with the special case of K_4. First, we embed the vertices of Γ at the corners of a regular tetrahedron. Then, we embed the edges of Γ so that each set of three edges whose vertices are the corners of a single face of the tetrahedron are now tangled as shown in Figure 6. The two dangling ends on each side in Figure 6 continue into an adjacent face with the same pattern. If we consider the knot formed by the three edges whose vertices are the corners of a single face (and ignore the other edges of Γ), we see that this cycle is embedded as the knot K illustrated in Figure 7.

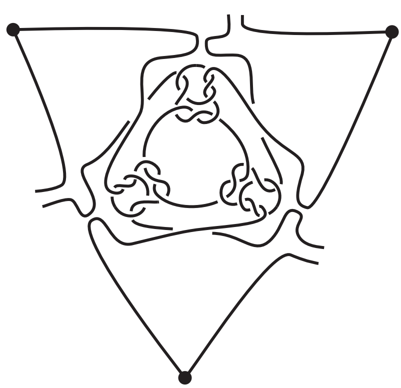

Figure 6: One face of the embedding Γ of K_4

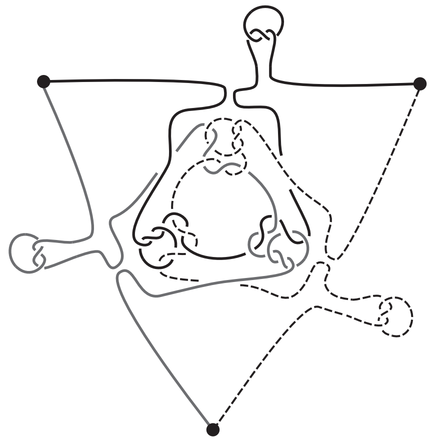

Figure 7: The knot K formed by three edges whose vertices are the corners of a single face. We indicate the three edges of the triangle with different types of lines.

Lemma 6.1 *The knot K in Figure 7 is noninvertible.*

Proof Observe that K is the connected sum of three trefoil knots together with the knot J illustrated in Figure 8. Suppose K is invertible. Then by the uniqueness of prime factorizations of oriented knots, J would also be invertible. Since J is the closure of the sum of three rational tangles, J is an algebraic knot. Thus the machinery of Bonahon and Siebenmann [1] can be used to show that J is noninvertible. It follows that K is noninvertible as well. □

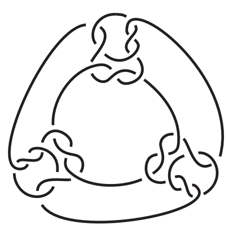

Figure 8: K is the connected sum of three trefoil knots together with the knot J illustrated here.

Proposition 6.2 *Let Γ be the embedding of K_4 in S^3 described above. Then* $\mathrm{TSG}_+(\Gamma) \cong A_4$*, and* $\mathrm{TSG}_+(\Gamma)$ *is induced by the group of rotations of a solid tetrahedron.*

Proof It follows from the Complete Graph Theorem that if A_4 is isomorphic to a subgroup of $\mathrm{TSG}_+(\Gamma)$, then $\mathrm{TSG}_+(\Gamma)$ is isomorphic to either A_4, S_4 or A_5. We will first show that $\mathrm{TSG}_+(\Gamma)$ contains a group isomorphic to A_4, and then that $\mathrm{TSG}_+(\Gamma)$ is not isomorphic to either S_4 or A_5.

We see as follows that Γ is setwise invariant under a group of rotations of a solid tetrahedron. The fixed point set of an order three rotation of a solid tetrahedron contains a single vertex of the tetrahedron and a point in the center of the face opposite that vertex. To see that Γ is invariant under such a rotation, we unfold three of the faces of the tetrahedron. The unfolded picture of Γ is illustrated in Figure 9. In order to recover the embedded graph Γ from Figure 9, we glue together the pairs of sides with corresponding labels. When we reglue these pairs, the three vertices labeled x become a single vertex. We can see from the unfolded picture in Figure 9 that there is a rotation of Γ of order three which fixes the point y in the center of the picture, together with the vertex x.

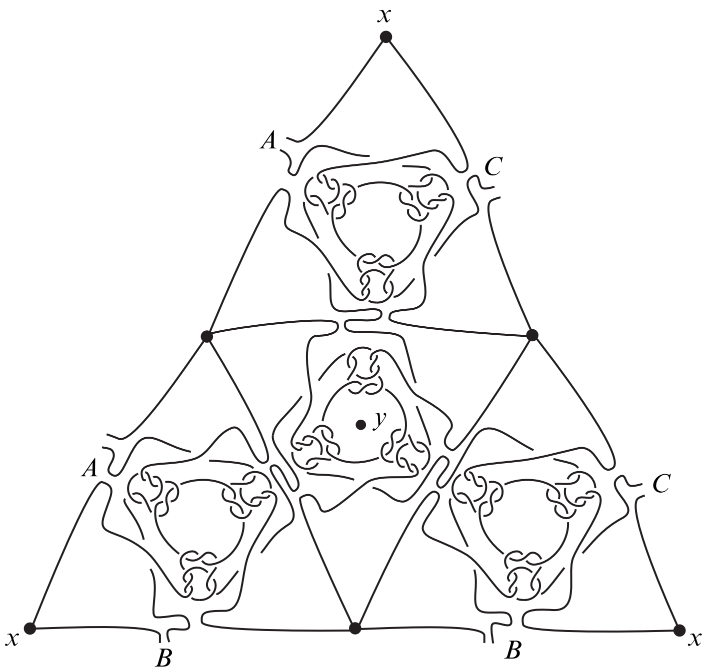

Figure 9: This unfolded view illustrates an order three symmetry of Γ.

The fixed point set of a rotation of order two of a tetrahedron contains the midpoints of two opposite edges. This rotation interchanges the two faces which are adjacent to each of these inverted edges. To see that Γ is invariant under such a rotation, we unfold the tetrahedron into a strip made up of four faces of the tetrahedron. The unfolded picture of Γ is illustrated in Figure 10. In order to recover the embedded graph Γ from Figure 10, we glue together pairs of sides with corresponding labels. When we glue these pairs, the two points labeled v are glued together. We can see from the unfolded picture in Figure 10 that there is a rotation of Γ of order two which fixes the point w that is in the center of the picture together with the point v. Thus $\mathrm{TSG}_+(\Gamma)$ contains a subgroup isomorphic to A_4.

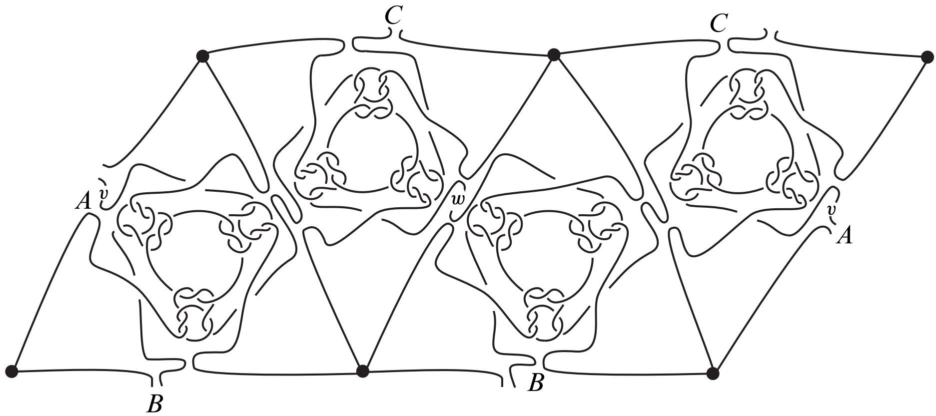

Figure 10: This unfolded view illustrates an order two symmetry of Γ.

Now assume that $S_4 \cong \mathrm{TSG}_+(\Gamma)$. Label the four vertices of Γ by the letters a, b, c, and d. Then there is a homeomorphism h of S^3 which leaves Γ setwise invariant while inducing the automorphism (ab) on its vertices. In particular, the image of the oriented cycle abc is the oriented cycle bac. Thus the simple closed curve in Γ with vertices abc is inverted by h. However, this simple closed curve is the knot K illustrated in Figure 7, and we proved in Lemma 6.1 that K is noninvertible. Therefore, $S_4 \ncong \mathrm{TSG}_+(\Gamma)$.

Finally, by Theorem 2.8, $A_5 \ncong \mathrm{TSG}_+(\Gamma)$, which completes the proof. □

Now we will show that for all $m > 4$ such that $m \equiv 0, 1, 4, 5, 8 \pmod{12}$, there is an embedding Γ of K_m in S^3 with $\mathrm{TSG}_+(\Gamma) \cong A_4$. The following Theorem from Flapan, Mellor and Naimi [4] will be used in the proof.

Subgroup Theorem [4] *Let Γ be an embedding of a 3–connected graph in S^3. Suppose that Γ contains an edge e which is not pointwise fixed by any nontrivial element of $\mathrm{TSG}_+(\Gamma)$. Then for every $H \leq \mathrm{TSG}_+(\Gamma)$, there is an embedding Γ' of Γ with $H = \mathrm{TSG}_+(\Gamma')$.*

Proposition 6.3 *Suppose that $m > 4$ and $m \equiv 0, 1, 4, 5, 8 \pmod{12}$. Then there is an embedding Γ of K_m in S^3 such that $\mathrm{TSG}_+(\Gamma) \cong A_4$.*

Proof We first consider the cases where $m \equiv 0, 4, 8, 12, 20 \pmod{24}$. Let G denote the finite group of orientation preserving isometries of the 1–skeleton τ of a regular tetrahedron. Recall from the proof of Proposition 4.1 that for each $k =$ 0, 4, 8, 12 or 20, we embedded K_{24n+k} as a graph Γ_k with vertices in the set $V_0 \cup V_4 \cup V_8 \cup V_{12}$ such that $\mathrm{TSG}_+(\Gamma_k) \cong S_4$ is induced by G. We will show that each Γ_k has an edge which is not pointwise fixed by any nontrivial element of G.

First suppose that $m = 24n + k$ where $n > 0$ and $k =$ 0, 4, 8, 12 or 20. Recall that V_0 contains $24n$ vertices none of which is fixed by any nontrivial element of G. Let e_0 be an edge of Γ_k with vertices in V_0. Then e_0 is not pointwise fixed by any nontrivial element of G. Hence by the Subgroup Theorem, there is an embedding Γ of K_m with $A_4 \cong \mathrm{TSG}_+(\Gamma)$.

Next we suppose that $n = 0$, and let Γ_8, Γ_{12}, and Γ_{20} denote the embeddings of K_8, K_{12}, and K_{20} given in the proof of Proposition 4.1. Let e_8 be an edge in Γ_8 whose vertices are not the endpoints of one of the gray arcs in Figure 2, let e_{12} be an edge in Γ_{12} whose vertices are not the endpoints of one of the gray arcs in Figure 3, and let e_{20} be an edge in Γ_{20} whose vertices are not the endpoints of one of the gray arcs in Figure 4. In each case, e_k is not pointwise fixed by any nontrivial element of G. Hence by the Subgroup Theorem, there is an embedding Γ of K_k with $A_4 \cong \mathrm{TSG}_+(\Gamma)$.

Next we consider the case where $m = 24n + 16$. Then $m = 12(2n + 1) + 4$. Let $H \cong A_4$ denote a finite group of orientation preserving isometries of S^3 which leaves a solid tetrahedron T setwise invariant. Then every element of H has nonempty fixed point set, and the only even order elements are involutions. Thus regardless of how we embed our vertices, hypothesis (5) of the Edge Embedding Lemma will be satisfied for H. Also, observe that no edge of the tetrahedron T is pointwise fixed by any nontrivial element of H. Let W_4 denote the vertices of T. Let B denote a ball which is disjoint from the fixed point set of any nontrivial element of H, and which is disjoint from its image under every nontrivial element of H. Choose $2n+1$ points in B (recall that we are not assuming that $n > 0$) and let W_0 denote the orbit of these points under H. We embed the vertices of $K_{12(2n+1)+4}$ as the points of $W_0 \cup W_4$. Since no pair of vertices in $W_0 \cup W_4$ are both fixed by a nontrivial element $h \in H$, it is easy to see that hypotheses (1)–(4) of the Edge Embedding Lemma are satisfied. Thus by applying the Edge Embedding Lemma we obtain an embedding Γ_{16} of K_{24n+16} which is setwise invariant under H. Now by Theorems 2.8 and 2.9, we know that $\mathrm{TSG}_+(\Gamma_{16}) \ncong A_5$ or S_4. Now it follows from the Complete Graph Theorem that $\mathrm{TSG}_+(\Gamma_{16}) \cong A_4$.

Thus we have shown that if $m \equiv 0, 4$ or $8 \pmod{12}$ and $m > 4$, then there is an embedding Γ of K_m in S^3 with $\mathrm{TSG}_+(\Gamma) \cong A_4$.

Next suppose that $m = 12n+1$ and $m \not\equiv 1 \pmod{60}$. Let the group H and the ball B be as in the above paragraph. Choose n points in B and let U_0 denote the orbit of these points under H. Let v denote one of the two points of S^3 which is fixed by every element of H. We embed the vertices of K_m as $U_0 \cup \{v\}$. Since no pair of vertices in $U_0 \cup \{v\}$ are both fixed by a nontrivial element $h \in H$, it is easy to check that the hypotheses of the Edge Embedding Lemma are satisfied. Now by the Edge Embedding Lemma together with Theorems 2.8 and 2.9 and the Complete Graph Theorem we obtain an embedding Γ of K_m such that $\mathrm{TSG}_+(\Gamma) \cong A_4$.

Similarly, suppose that $m = 12n+5$ and $m \not\equiv 5 \pmod{60}$. Let H, U_0, W_4, and v be as above. We embed the vertices of K_m as $U_0 \cup W_4 \cup \{v\}$. The arcs required by hypothesis (2) of the Edge Embedding Lemma are highlighted in gray in Figure 11. Now again by the Edge Embedding Lemma together with Theorems 2.8 and 2.9 and the Complete Graph Theorem we obtain an embedding Γ of K_m such that $\mathrm{TSG}_+(\Gamma) \cong A_4$.

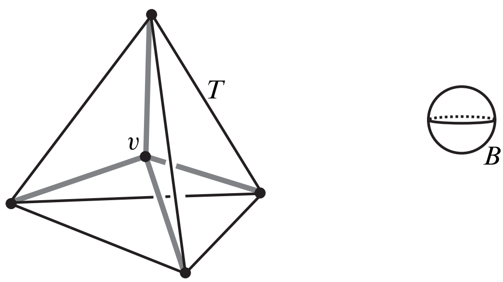

Figure 11: The vertices of $W_4 \cup \{v\}$ are indicated by black dots. The arcs required by hypothesis (3) are highlighted in gray.

Next suppose that $m = 60n+1$ or $m = 60n+5$ where $n > 0$. Let G_1 denote the group of orientation preserving symmetries of a regular solid dodecahedron and let G_5 denote the group of orientation preserving symmetries of a regular 4–simplex. We first embed K_{60n+1} and K_{60n+5} as the graphs Γ_1 and Γ_5 respectively given in the proof of Proposition 5.1 such that $\mathrm{TSG}_+(\Gamma_k) \cong A_5$ is induced by G_k, where $k = 1, 5$. Since $n > 0$, we can choose an edge e_0 of Γ_k both of whose vertices are in V_0. Then e_0 is not pointwise fixed by any nontrivial element of G_k . Hence by the Subgroup Theorem, we obtain an embedding Γ of K_m such that $\mathrm{TSG}_+(\Gamma) \cong A_4$.

Finally, let $m = 5$. Let μ denote an embedding of the 1–skeleton of a regular solid tetrahedron T so that the edges of μ each contain an identical trefoil knot. Let Γ denote these vertices and edges together with a vertex at the center of T which is connected via unknotted arcs to the other vertices of T (see Figure 12). We choose Γ so that it is setwise invariant under a group of orientation preserving isometries of T. Thus $\mathrm{TSG}_+(\Gamma)$ contains a subgroup isomorphic to A_4. Since Γ is an embedding of K_5, by Theorem 2.8 we know that $\mathrm{TSG}_+(\Gamma) \ncong S_4$. Furthermore, any homeomorphism of (S^3, Γ), must take each triangle which is the connected sum of 3 trefoil knots to a triangle which also is the connected sum of 3 trefoil knots. Thus $\mathrm{TSG}_+(\Gamma)$ must leave μ setwise invariant. Since μ is an embedding of K_4, $\mathrm{TSG}_+(\Gamma)$ induces a faithful action on K_4. However, A_5 cannot act faithfully on K_4. Thus $\mathrm{TSG}_+(\Gamma) \ncong A_5$. Now it follows from the Complete Graph Theorem that $\mathrm{TSG}_+(\Gamma) \cong A_4$.

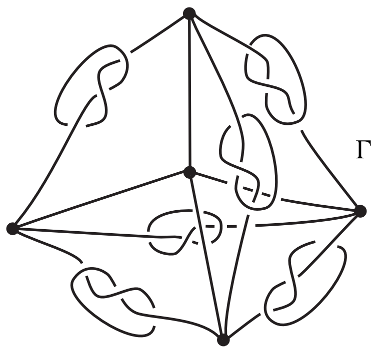

Figure 12: An embedding Γ of K_5 such that $\mathrm{TSG}_+(\Gamma) = A_4$

The above four paragraphs together show that if $m \equiv 1, 5 \pmod{12}$ and $m > 4$, then there is an embedding Γ of K_m in S^3 with $\mathrm{TSG}_+(\Gamma) \cong A_4$. $\square$

The following theorem summarizes our results on when a complete graph can have an embedding whose topological symmetry group is isomorphic to A_4.

A_4 Theorem *A complete graph K_m with $m \geq 4$ has an embedding Γ in S^3 such that $\mathrm{TSG}_+(\Gamma) \cong A_4$ if and only if $m \equiv 0, 1, 4, 5, 8 \pmod{12}$.*

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